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STABILIZATION OF SOLUTIONS OF A DEGENERATE NONLINEAR DIFFUSION --ETC(U)
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ABSTRACT

The central results concern the initial-value problem for $u_t = (u^m)_{xx} + u(1-u)(u-a)$, $-L < x < L$ and $t > 0$, under Dirichlet conditions at $x = \pm L$. Here $m > 1$ and $0 < a < (m+1)/(m+3)$. The equilibrium solutions of this problem are determined for each $L > 0$ and it is shown that the ω-limit set $\omega(u_0)$ of an initial datum u_0 with values in $[0,1]$ consists of a connected set of equilibria. This is used to determine some domains of attraction of isolated equilibria. A novel feature of the results is that for large L there are multiple parameter families of equilibria.

A second part of the paper gives a self-contained development of existence, uniqueness, maximum principles, and continuous dependence on data for more general equations $u_t = \eta(u)_{xx} + f(u)$. The results are employed in proofs of some of the theorems referred to above.

Interest in these questions is stimulated by the occurrence of such models in science, e.g. in fluid flow in porous media and biology.

AMS (MOS) Subject Classifications: 35K55, 35K65

Key Words: nonlinear diffusion, biological models, asymptotic behaviour, equilibrium solutions, flow in porous media.

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STABILIZATION OF SOLUTIONS OF A DEGENERATE NONLINEAR

DIFFUSION PROBLEM

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Introduction

In this paper we are primarily concerned with the large time behaviour of nonnegative solutions of the initial-boundary value problem

$$\text{(I)} \qquad \begin{cases} u_t = (u^m)_{xx} + f(u) & \text{in} \quad (-L,L) \times R^+ , \\ u(\pm L,t) = 0 & \text{in} \quad R^+ , \\ u(x,0) = u_0(x) & \text{in} \quad [-L,L] , \end{cases}$$

where $m > 1$ is a parameter, f is locally Lipschitz continuous, $f(0) = 0$, and u_0 is bounded. Problems of this form arise in a number of areas of science; for instance, in models for gas or fluid flow in porous media [2] and for the spread of certain biological populations [13, 16].

This paper is divided into two part. In Part I we consider what may be called the motivating example, Problem I^*, which consists of Problem I with the special choice

$$(1) \qquad f(u) = u(1-u)(u-a)$$

for suitably restricted parameters a. We begin by describing in detail the set $\mathcal{E} = \mathcal{E}(L)$ of nonnegative equilibrium solutions of Problem I^*. Clearly $\mathcal{E}(L)$ contains the trivial solution $u = 0$ for all $L > 0$. Write

$$\mathcal{E}^*(L) = (L)\setminus\{0\} .$$

In the description of $\mathcal{E}^*(L)$ there are two critical parameter values L_0 and L_1 with $0 < L_0 < L_1 < +\infty$. We show that:

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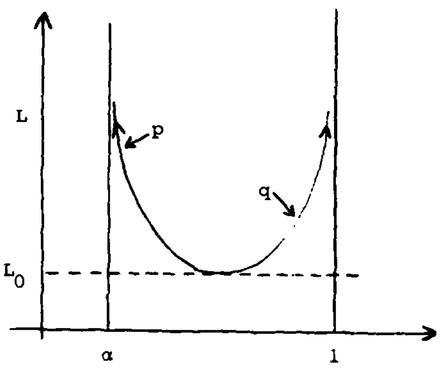
(i) $E^*(L) = \phi$ for $0 < L < L_0$

(ii) $E^*(L_0)$ consists of one isolated positive solution

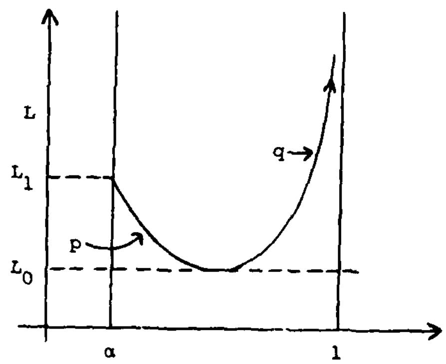
(iii) For $L_0 < L < L_1$, $E^*(L)$ consists of two isolated positive solutions, p and q, with $p < q$ on $(-L,L)$.

(iv) For $L > L_1$, N a positive integer, and $NL_1 < L < (N+1)L_1$, $E^*(L)$ consists of one isolated positive solution q and N j-parameter families $P_j(L)$, $j = 1,\ldots,N$, of nonnegative solutions. If $L = (N+1)L_1$, $E^*(L)$ contains one additional element.

Recently Smoller and Wasserman [18] studied $E(L)$ *for Problem* I^* *in the case* $m = 1$. In contrast to the result stated above, they find that $L_1 = +\infty$ for $m = 1$. This situation is summarized in the two diagrams in Figure 1. These diagrams indicate the general behaviour of possible values of L plotted against u_{max} (the maximum of u) for $u \in E^*(L)$.



(a) $m = 1$



(b) $m > 1$

Figure 1

Having described the equilibrium set $E(L)$ for Problem I^* we next turn our attention to the question of stability of the various equilibria. As in the case $m = 1$, it turns out that both the trivial solution and the large positive solution q are stable. To establish this fact we apply a stabilization theorem proved in Part II in the general setting of Problem I.

We begin Part II by proving various basic existence, uniqueness, comparison and regularity theorems for Problem I. None of these results are entirely new, but we know of no place in the literature where they are conveniently collected. Further results of Part II - in particular, the stabilization theorem - provide us with a complete metric space X of functions on $(0,1)$ in which the orbits of Problem I^* are precompact. Moreover, if $0 \le u_0 \le 1$ and $u(t,u_0)$ is the solution of Problem I^* at time t, then the ω-limit set

$$\omega(u_0) = \{w \in X : \text{there exists a sequence } \{t_n\}, t_n \to \infty \text{ such that } u(t_n,u_0) \to w \text{ in } X\} \tag{2}$$

is contained in $E(L)$. For $n = 1$ this was proved by Chafee and Infante [7].

If $E(L)$ consists of isolated points only, as in the case of Problem I^* and $0 < L \le L_1$, we obtain that $u(t,u_0)$ converges to a limit in $E(L)$ as $t \to \infty$ (since $\omega(u_0)$ is connected). If $E(L)$ contains continua of solutions, as in the case for Problem I^* and $L > L_1$, then no such statement has been proved. However, if one can find a closed invariant subset $K \subset X$ such that $K \cap E(L)$ is discrete, then for each $u_0 \in K$, $u(t,u_0)$ converges to some point of $K \cap E(L)$ as $t \to \infty$. The stability of the trivial solution and the large positive solution q of Problem I^* for $L > L_0$ are proved by exhibiting suitable invariant sets K.

Part I Problem I^*

In this part we shall consider Problem I, assuming throughout that

$$f(u) = u(1-u)(u-a),\ 0 < a < (m+1)/(m+3) \quad . \tag{1.1}$$

First $E(L)$ is studied and then, calling upon results from Part II below, $\omega(u_0)$ is determined for various choices of L and u_0.

Section 1 Equilibrium Solutions

We are interested only in nonnegative solutions. A function $v : [-L,L] \to R^+ = [0,\infty)$ is an equilibrium solution of Problem I when it is a solution of the problem

$$\text{(II)} \qquad \begin{cases} (v^m)'' + f(v) = 0 & \text{in} \quad (-L,L) \quad , \\ v(\pm L) = 0 \quad . \end{cases}$$

While we will write (II) in the above form, $v \geq 0$ is called a solution of (II) exactly when $w = v^m$ is a classical solution of $w'' + f(w^{1/m}) = 0$, $w(\pm L) = 0$. Clearly $v \equiv 0$ is always a solution of Problem II. As we shall show below, there are also nontrivial solutions provided that L is sufficiently large.

Suppose v is a <u>positive solution</u> of Problem II, i.e. it is a solution and $v > 0$ on $(-L,L)$. Then there exists a $\zeta \in (-L,L)$ such that $0 < v(x) \leq v(\zeta)$ for $x \in (-L,L)$ and, clearly, $v'(\zeta) = 0$. Conversely, let us seek conditions on $\zeta \in (-L,L)$ and $\mu \in R^+$ which guarantee that the solution of the initial value problem

$$\text{(II')} \qquad \begin{cases} (v^m)'' + f(v) = 0 \\ v(\zeta) = \mu, \; v'(\zeta) = 0 \end{cases}$$

is also a positive solution of Problem II.

If $\mu = 1$ then $v \equiv 1$ is the unique solution of (II') (recall (1.1) - $f(1) = 0$). If $\mu > 1$, then $f(v) < 0$ for $v > 1$ implies that any solution of (II') is convex on its domain of definition and hence cannot satisfy $v(\pm L) = 0$. Thus (II') has no solutions satisfying the boundary conditions unless $\mu < 1$. Consequently, we consider only $\mu \in (0,1)$.

To solve Problem II' we integrate the equation in the usual way. Multiply the equation by $(v^m)'$, and integrate the result using the initial conditions to find

$$\text{(1.2)} \qquad \frac{1}{2} (v^m)'^2 + mF(v) = mF(\mu)$$

where

$$F(v) = \int_0^v s^{m-1} f(s)ds \quad .$$

More explicitly,

$$(1.3) \qquad F(v) = -\frac{1}{m+3} v^{m+1} G(v)$$

where

$$G(v) = v^2 - (1+a)\frac{m+3}{M+2} v + a\frac{m+3}{m+1} \quad .$$

Since $f > 0$ on $(a,1)$, F is strictly increasing on $(a,1)$. Thus, if $\mu > a$, we can integrate (1.2) to obtain

$$(1.4) \qquad \sqrt{\frac{m}{2}} \int_v^\mu \frac{\eta^{m-1}}{\sqrt{F(\mu)-F(\eta)}} d\eta = |\zeta - x| \quad .$$

The integrand in (1.4) has a singular point at $\eta = \mu$, but $F(\mu) - F(\eta) > \delta(\mu-\eta)$ for some $\delta > 0$ and η near μ so the singularity is integrable. Equation (1.4) defines v implicitly as a function of $|\zeta-x|$ so long as $v < \mu$.

If $F(\mu) < 0$, then there exists a unique $\nu \in (0,a)$ such that $F(\nu) = F(\mu)$ and $F(\eta) < F(\mu)$ for $\eta \in (\nu,\mu)$. In this case, (1.4) represents a periodic solution of $(v^m)'' + f(v) = 0$ whose values lie in $[\nu,\mu]$. Thus, in order that (1.4) represent a positive solution of Problem II it is necessary that $F(\mu) > 0$.

The sign of F is determined by the sign of G and one checks that G has a unique root $\alpha \in (a,1)$ if and only if $G(1) < 0$ or

$$(1.5) \qquad 0 < a < \frac{m+1}{m+3} \quad .$$

In particular, if (1.5) holds then

$$F < 0 \text{ on } (0,\alpha) \text{ and } F > 0 \text{ on } (\alpha,1) \quad ,$$

whence we may restrict our attention to the range $\alpha \leq \mu < 1$.

For $\mu \in [\alpha,1)$ we have $F(\eta) < F(\mu)$ for all $\eta \in (0,\mu)$. Thus we can extend the integration in (1.4) down to $v = 0$. Define

$$(1.6) \qquad \lambda(\mu) = \sqrt{\frac{m}{2}} \int_0^{\mu} \frac{\eta^{m-1}}{\sqrt{F(\mu)-F(\eta)}}\, d\eta, \quad \alpha \leq \mu < 1 \quad .$$

If $\mu = \alpha$ the integrand in (1.6) may have a second singularity at $\eta = 0$. However, $-F(\eta) > \delta^2 \eta^{m+1}$ for some $\delta > 0$ and $\eta > 0$ near 0, so $\eta^{m-1}(-F(\eta))^{-1/2} < \delta^{-1}\eta^{1/2\,(m-3)}$ near $\eta = 0$. Since $m > 1$ this singularity is integrable and λ is well-defined on $[\alpha,1)$.

For a positive solution v of Problem II, $v = 0$ only at $\pm L$. Therefore

$$\lambda(\mu) = |\zeta - L| = |\zeta + L|$$

from which we conclude that $\zeta = 0$. To summarize, we have proven the following result:

<u>Proposition 1</u> Suppose $0 < a < (m+1)/(m+3)$. Then v is a positive solution of Problem II if and only if

$$\sqrt{\frac{m}{2}} \int_{v(x)}^{\mu} \frac{\eta^{m-1}}{\sqrt{F(\mu)-F(\eta)}}\, d\eta = |x| \quad \text{for} \quad |x| < L \quad ,$$

where $\mu \in [\alpha,1)$ and $L \in R^+$ are related by the equation

$$(1.7) \qquad \lambda(\mu) = L$$

and α is the unique root of F in $(a,1)$.

In view of (1.7), there is a positive solution of Problem II for a given interval $(-L,L)$ if and only if L is in the range of λ, i.e. $L \in \lambda([\alpha,1))$. When $L = \lambda(\mu)$ we write $v(x,\mu)$ for the corresponding positive solution. The multiplicity of these positive solutions is the same as the multiplicity of the roots of $\lambda(\mu) = L$, which is determined by the shape of the graph of λ. Our next result shows that the graph of $\lambda(\mu)$ always has the general features indicated in Figure 1.

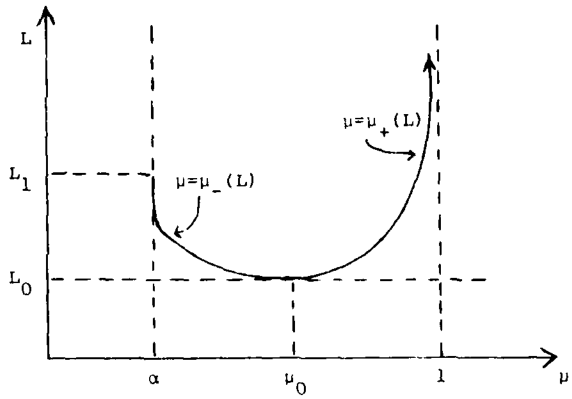


Figure 2

<u>Proposition 2</u>: (i) $\lambda \in C[\alpha,1) \cap C^1(\alpha,1)$

(ii) $\lambda(\mu) \to +\infty$ and $\lambda'(\mu) \to +\infty$ as $\mu \uparrow 1$.

(iii) $\lambda'(\mu) \to -\infty$ as $\mu \downarrow \alpha$

(iv) $\lambda'(\mu)$ has a unique root $\mu_0 \in (\alpha,1)$.

Before proving Proposition 2, *we shall make a few remarks* about its interpretation. Define $L_0 = \lambda(\mu_0)$ and $L_1 = \lambda(\alpha)$. Clearly $L_0 > 0$ and by (ii) $\lambda([\alpha,1]) = [L_0, +\infty)$. Moreover

$$\lambda(\mu) = L \text{ has } \begin{cases} \text{no solutions for } 0 < L < L_0 , \\ \text{one solution for } L = L_0 \text{ for } L > L_1 , \\ \text{two solutions for } L_0 < L \leq L_1 . \end{cases}$$

It is interesting to note the dependence of L_1 on m. Let us write $F = F(\eta;m)$, $\alpha = \alpha(m)$ and $L_1 = L_1(m)$. Then

$$L_1(m) = \sqrt{\frac{m}{2}} \int_0^{\alpha(m)} \frac{\eta^{m-1}}{\sqrt{-F(\eta;m)}} \, d\eta \quad .$$

If $a \in (0,1/2)$ then $\alpha(m)$ is defined and continuous for $m > 1$. Moreover, as $m \to 1$, $F(\eta;m) \to F(\eta;1)$ and the singularity of $\left(\eta\sqrt{F(\alpha(1);1) - F(\eta;1)}\right)^{-1} = \left(\eta\sqrt{F(-\eta;1)}\right)^{-1}$ at $\eta = 0$ is not integrable. It follows from Fatou's lemma that $\lim\limits_{m\downarrow 1} L_1(m) = \infty$. Thus the nonexistence of the small positive solution on sufficiently large intervals is due to the nonlinearity of the diffusion.

<u>Proof of Proposition 2</u>

Write

$$\Lambda(\mu) = \sqrt{\frac{2}{m}}\, \lambda(\mu) = \int_0^{\mu} \frac{\eta^{m-1}}{\sqrt{F(\mu)-F(\eta)}} \, d\eta$$

and use the change of variables $\tau = \eta/\mu$ to obtain the expression

$$\Lambda(\mu) = \mu^m \int_0^1 \frac{\tau^{m-1}}{\sqrt{F(\mu)-F(\tau\mu)}} \, d\tau \quad .$$

Formally differentiating the integral yields

$$\Lambda'(\mu) = \frac{m}{\mu}\Lambda(\mu) - \frac{\mu^m}{2}\int_0^1 \tau^{m-1} \frac{(F'(\mu)-\tau F'(\tau\mu))}{(F(\mu)-F(\tau\mu))^{3/2}} \, d\tau \quad . \tag{1.8}$$

For $\mu \in (\alpha,1)$ we have $F'(\mu) = \mu^{m-1} f(\mu) \neq 0$ and it is not difficult to verify that the integral in (1.8) is convergent, the equation is valid and, indeed, $\Lambda' \in C(\alpha,1)$. If we set

$$\theta(\eta) = 2mF(\eta) - \eta^m f(\eta)$$

then Λ' can also be written in the form

$$\Lambda'(\mu) = \frac{1}{2\mu}\int_0^{\mu} \frac{\theta(\mu)-\theta(\eta)}{(F(\mu)-F(\eta))^{3/2}} \, \eta^{m-1} d\eta \quad . \tag{1.9}$$

To study the behaviour of $\Lambda(\mu)$ as $\mu \downarrow \alpha$, write

$$\Lambda(\mu) = \left(\int_0^\alpha + \int_\alpha^\mu\right) \frac{\eta^{m-1}}{\sqrt{F(\mu)-F(\eta)}}\, d\eta \equiv I_1(\mu) + I_2(\mu) \quad .$$

For $\eta < \alpha < \mu$,

$$\frac{\eta^{m-1}}{\sqrt{F(\mu)-F(\eta)}} \leq \frac{\eta^{m-1}}{\sqrt{-F(\eta)}}$$

and we have already noted that the right-hand side is integrable on $[0,\alpha]$, so

$$I_1(\mu) \to \int_0^\alpha \frac{\eta^{n-1}}{\sqrt{-F(\eta)}}\, d\eta = \Lambda(\alpha) \quad \text{as} \quad \mu \downarrow \alpha \quad .$$

On the other hand, by previous remarks, on each compact subset of $\alpha \leq \eta \leq \mu < 1$, $\eta^{m-1}/\sqrt{F(\mu)-F(\eta)}$ is dominated by a multiple of $(\mu-\eta)^{-1/2}$, so $I_2(\mu) \to 0$ as $\mu \downarrow \alpha$ and it follows that $\Lambda \in C([\alpha,1))$.

Since $\theta(\alpha) = -\alpha^m f(\alpha) < 0$ we can choose $\gamma > 0$ such that $\theta(\mu) < \theta(\alpha)/2 < 0$ for $\mu \in [\alpha, \alpha+\gamma)$. In addition, since $\theta(0) = 0$ we can choose $\delta > 0$ such that $|\theta(\eta)| < -\theta(\alpha)/4$ for $\eta \in [0,\delta)$. Thus, in particular, $\mu \in [\alpha, \alpha+\gamma)$ and $\eta \in [0,\delta)$ imply

$$\theta(\mu) - \theta(\eta) < \frac{\theta(\alpha)}{4} < 0 \quad .$$

Write

$$\Lambda'(\mu) = \frac{1}{2\mu}\left\{\int_0^\delta + \int_\delta^\mu\right\} \frac{\theta(\mu)-\theta(\eta)}{(F(\mu)-F(\eta))^{3/2}}\, \eta^{m-1}\, d\eta \equiv J_1(\mu) + J_2(\mu) \quad .$$

Arguments like those above show that $J_2(\mu)$ remains bounded as $\mu \downarrow \alpha$. For $\mu \in (\alpha, \alpha+\gamma)$ we have

$$J_1(\mu) < \frac{\theta(\alpha)}{8\mu} \int_0^\delta \frac{\eta^{m-1}}{(F(\mu)-F(\eta))^{3/2}}\, d\eta \equiv \frac{\theta(\alpha)}{8\mu} J^*(\mu) \quad .$$

As $\mu \downarrow \alpha$, the integrand of $(J^*(\mu)$ converges pointwise to $\eta^{m-1}(-F(\eta))^{-3/2}$ which behaves like $\eta^{m-1}(\eta)^{-3/2(m+1)} = \eta^{-1/2(m+5)}$ near $\eta = 0$ and so is not integrable. Thus Fatou's lemma yields $J^*(\mu) \to +\infty$ (and so $J_1(\mu)$, $\Lambda'(\mu) \to -\infty$) as $\mu \downarrow \alpha$.

To show that $\Lambda(\mu) \to \infty$ and $\Lambda'(\mu) \to \infty$ as $\mu \uparrow 1$ we argue similarly. First, the integrand in $\Lambda(\mu)$ tends to $\eta^{m-1}/\sqrt{F(1)-F(\eta)}$. Now $F'(1) = f(1) = 0$ and $F''(1) < 0$, so this has a nonintegrable singularity at 1 and it follows that $\Lambda(\mu) \to +\infty$ as $\mu \uparrow 1$. The integrand in $\Lambda'(\mu)$ also tends to a nonintegrable limit since $\theta'(1) > 0$, and it follows that $\Lambda'(\mu) \to +\infty$ as $\mu \to 1$.

We now turn to the proof of (iv). For this we follow closely the proof given in the case $m = 1$ by Smoller and Wasserman [18].

To begin with, we note several properties of the function θ. They are all proved by elementary calculations using the explicit formulas for f and F. We shall omit the details.

A. There is a $\mu_1 \in (\alpha,1)$ such that $\theta(\eta) < 0$ on $(0,\mu_1)$ and $\theta(\eta) > 0$ on $(\mu_1,1]$.

B. There is a $\mu_2 \in (0,\mu_1)$ such that $\theta'(\eta) < 0$ on $(0,\mu_2)$ and $\theta'(\eta) > 0$ on $(\mu_2,1]$.

C. There exists a $\mu_3 \in (0,\mu_2)$ such that $(\eta\theta'(\eta))' < 0$ on $(0,\mu_3)$ an $(\eta\theta'(\eta))' > 0$ on $(\mu_3,1]$.

It follows from properties A and B that

$$\Lambda'(\mu) > 0 \quad \text{on} \quad (\mu_1,1)$$

and, if $\alpha < \mu_2$,

$$\Lambda'(\mu) < 0 \quad \text{on} \quad (\alpha,\mu_2) \quad .$$

Thus we need only consider Λ' for $\max(\alpha,\mu_2) \leq \mu \leq \mu_1$.

To proceed we need to examine Λ''. For this purpose, let

$$(\delta_1 h)(\eta) = h(\mu) - h(\eta), \qquad 0 \le \eta < \mu$$

$$(\delta_2 h)(\eta) = \mu h(\mu) - \eta h(\eta), \qquad 0 \le \eta < \mu$$

and

$$(\delta_3 h)(\eta) = \mu^m h(\mu) - \eta^m h(\eta), \qquad 0 \le \eta < \mu \quad .$$

A computation similar to the one yielding (1.9) produces

$$\Lambda''(\mu) = \frac{1}{2\mu^2} \int_0^\mu \frac{(\delta_1 F)(\delta_2 \theta') - \frac{3}{2}(\delta_1 \theta)(\delta_3 f)}{(\delta_1 F)^{5/2}} \eta^{m-1}\, d\eta + \frac{m-1}{\mu} \Lambda'(\mu) \quad . \tag{1.10}$$

Adding $(K - \frac{m-1}{\mu})\Lambda'(\mu)$ to both sides of (1.10) we obtain

$$\Lambda''(\mu) + (K - \frac{m-1}{\mu})\Lambda'(\mu) = \frac{1}{2\mu^2} \int_0^\mu \frac{(\delta_1 F)(\delta_2 \theta') + \delta_1 \theta (K\mu\delta_1 F - \frac{3}{2}\delta_3 f)}{(\delta_1 F)^{5/2}} \eta^{m-1}\, d\eta \quad .$$

Observe next that $\delta_1 \theta = 2m\delta_1 F - \delta_3 f$ so that $K\mu\delta_1 F - \frac{3}{2}\delta_3 f = (K\mu - 3m)\delta_1 F + \frac{3}{2}\delta_1 \theta$. Thus, if we choose $K = 3m/\mu$ we arrive at the expression

$$\Lambda''(\mu) + \frac{2m+1}{\mu} \Lambda'(\mu) = \frac{1}{2\mu^2} \int_0^\mu \frac{\frac{3}{2}(\delta_1 \theta)^2 + (\delta_1 F)(\delta_2 \theta')}{(\delta_1 F)^{5/2}} \eta^{m-1}\, d\eta \quad .$$

In view of properties B and C above, we have

$$\delta_2 \theta' = \mu\theta'(\mu) - \eta\theta'(\eta) > 0 \quad \text{for} \quad \max(\alpha, \mu_2) < \mu \le 1$$

and it follows that

$$\Lambda''(\mu) + \frac{2m+1}{\mu} \Lambda'(\mu) > 0 \quad \text{for} \quad \max(\alpha, \mu_2) \le \mu \le \mu_1 \quad . \tag{1.11}$$

As noted above, $\Lambda'(\mu) < 0$ for $\mu < \max(\alpha, \mu_1)$ and $\Lambda'(\mu) > 0$ for $\mu > \mu_1$. Therefore Λ' has at least one zero in the interval $J = [\max(\alpha, \mu_2), \mu_1]$. The relation (1.11) implies $\Lambda''(\mu) > 0$ at any such zero and therefore there can be at most one, completing the proof.

__Remark:__ As the proof shows, the result (iv) is dependent only on the properties A, B, C of θ.

Propositions 1 and 2 provide a complete characterization of the set of positive solutions of Problem II. For $L_0 \le L$ let $\mu_+(L)$ denote the largest solution of $L = \lambda(\mu)$ and for $L_0 \le L \le L_1$ let $\mu_-(L)$ be the smallest solution (so $\mu_+(L_0) = \mu_-(L_0)$). We distinguish the following cases

__$0 < L < L_0$.__ There are no positive solutions.

__$L = L_0$.__ There is a unique positive solution $v(\cdot,\mu_+(L_0))$.

__$L_0 < L \le L_1$.__ There are two positive solutions $p(\cdot,L) = v(\cdot,\mu_-(L))$ and $q(\cdot,L) = v(\cdot,\mu_+(L))$ with $p < q$ everywhere on $(-L,L)$.

__$L > L_1$.__ There is one positive solution $q(\cdot,L) = v(\cdot,\mu_+(L))$.

Since $v(\cdot,\mu)$ depends continuously on μ and $\mu_\pm(L)$ are continuous on their domains, p and q are continuous functions of L on their domains.

We now show that $v(\cdot,\alpha) = v(\cdot,\mu_-(L_1))$ generates families of nonnegative solutions of Problem II on intervals $(-L,L)$ with $L > L_1$. For $\mu \in (\alpha,1]$ we have $F(\mu) > 0$ so that, according to (1.2) $(v^m)'(\pm\lambda(\mu),\mu) \neq 0$. However, $F(\alpha) = 0$, so $(v^m)'(\pm\lambda(\alpha),\alpha) = (v^m)'(\pm L_1,\alpha) = 0$. It follows that $v(x,\alpha)$ extended as 0 for $L \ge |x| \ge L_1$ is a solution of Problem II for $L > L_1$ and so is

$$r(x;h) = \begin{cases} v(x-h;\alpha) & \text{for } |x-h| \le L_1 \\ 0 & \text{for } |x-h| \ge L_1 \end{cases}$$

provided $|h| \le L - L_1$. More generally, we may piece several such solutions together if their supports are disjoint. Let N be a positive integer and $L \ge NL_1$. For each N-vector $\zeta = (\xi_1,\ldots,\xi_N)$ which satisfies

$$(1.12)\quad -L \le \xi_1 - L_1,\ \zeta_i + L_1 \le \xi_{i+1} - L_1,\ i = 1,\ldots,N-1 \text{ and } \zeta_N + L_1 \le L\ ,$$

the function

$$r(x;\xi) = \begin{cases} v(x-\xi_i,\alpha) & \text{for } |x-\xi_i| \leq L_1 , \\ 0 & \text{if } |x-\xi_i| > L_1 \text{ for } i = 1,\dots,N \end{cases}$$

is a nonnegative solution of Problem II. We shall use $P_N(L)$ to denote the collection of functions $v(\cdot,\xi)$ where $\xi \in R^N$ satisfies (1.12).

Clearly a nonnegative solution of Problem II is either positive or belongs to some $P_N(L)$. We thus have:

<u>Proposition 3</u> For $L > L_1$ let N be the integral part $[L/L_1]$ of L/L_1. Then with $P(L) = \bigcup_{j=1}^{N} P_j(L)$ we have

$$E^*(L) = \{q(\cdot,L)\} \cup P(L) .$$

<u>Remark</u>. If $L/L_1 > [L/L_1] = N$ then $P_N(L)$ is a true N-parameter family, while if $N = L/L_1$, $P_N(L)$ contains only $r(x;\xi)$, $\xi_i = (i-1)L_1 + L_1/2$. Combining Propositions 1.1, 1.2 and 1.3 we arrive at the complete description of $E(L)$ given below:

<u>Theorem 4</u>.

$$E(L) = \begin{cases} \{0\} & \text{for } 0 < L < L_0 , \\ \{0,\ p(\cdot;L),\ q(\cdot;L)\} & \text{for } L_0 \leq L \leq L_1 , \\ \{0,\ q(\cdot,L)\} \cup P(L) & \text{for } L_1 < L . \end{cases}$$

<u>Section 2</u>. Stability Theory

We now turn to the question of the large time behaviour of the solution of the initial-bounday value problem

$$(I^*) \qquad \begin{cases} u_t = (u^m)_{xx} + f(u) & \text{in } (-L,L) \times R^+ \\ u(\pm L,t) = 0 & \text{in } R^+ \\ u(\cdot,0) = u_0 & \text{in } (-L,L) \end{cases}$$

where f is given by (1.1) and

$$(1.13) \qquad u_0 \in L^\infty(-L,L),\ 0 \leq u_0 \leq 1 \ \text{ a.e.}$$

In what follows we shall, for convenience, write

$$\Omega = (-L,L),\ \overline{Q_T} = \Omega \times (0,T],\ Q = \Omega \times R^+ \quad .$$

Definition. A solution u of Problem I^* on $[0,\infty)$ is a function $u : [0,\infty) \to L^1(\Omega)$ with the properties

(i) $u \in C([0,\infty) : L^1(\Omega)) \cap L^\infty(Q_T)$ for $T > 0$.

(ii) $\int_\Omega u(T)\varphi(T) - \iint_{Q_T} (u\varphi_t + u^m\varphi_{xx}) = \int_\Omega u_0\varphi(0) + \iint_{Q_T} f(u)\varphi$

for all $T > 0$ and $\varphi \in C^2(\overline{Q})$ such that $\varphi \geq 0$ in Q and $\varphi = 0$ at $x = \pm L$.

A subsolution (supersolution) of Problem I^* on $[0,\infty)$ is a function satisfying (i) and (ii) with equality replaced by $\leq$ (respectively, $\geq$).

Theorem 5 (Existence and comparison)

(i) If (1.13) holds Problem I^* has a unique solutien u on $[0,\infty)$ and $0 \leq u \leq 1$ a.e.

(ii) If u is a subsolution and $\hat{u}$ is a supersolution of Problem I^* then $u \leq \hat{u}$ a.e. on Q.

Theorem 5 is a consequence of more general results in Part II (Theorem 12 and 13). We will denote the solution of Problem I^* with the initial value u_0 by $u(t,u_0)$. Let X be the complete metric space with the metric d given by

$$(1.14)\qquad \begin{cases} X = \{u \in L^\infty(\Omega) : 0 \leq u \leq 1 \text{ a.e. and } u^m \in H^1(\Omega)\} \ , \\ d(u,v) = \|u-v\|_{L^1(\Omega)} + \|(u^m - v^m)_x\|_{L^2(\Omega)} \end{cases}$$

Recall the definition of $\omega(u_0)$ (Introduction, eq. (2)). A stablization theorem (Theorem 18) is proved in Part II which applies to Problem I^* to yield:

Theorem 6 (Stabilization): Let u_0 satisfy (1.13). Then $\{u(t,u_0) : t \geq 1\}$ is a compact subset of X, $\omega(u_0)$ is nonempty and connected in X and $\omega(u_0) \subset E(L)$.

We will use these results to show the stability of the equilibrium solutions $v = 0$ and $v = q$. For this purpose we introduce the notion of sub-and supersolutions of Problem II,

$$(v^m)'' + f(v) = 0 \ \text{ in } (-L,L),\ v(\pm L) = 0 \quad .$$

A (weak) subsolution of Problem II is a function $v \in C([-L,L])$ for which $\int_\Omega(\varphi'' v^m + \varphi f(v))dx \geq 0$ for all $\varphi \in C^2(\bar\Omega)$, $\varphi \geq 0$ and $\varphi(\pm L) = 0$ and $v(\pm L) \leq 0$. A (weak) supersolution is defined by reversing the inequality and requiring $v(\pm L) \geq 0$.

Let $\underline{v}$ and $\bar{v}$ be respectively a sub-and a supersolution of Problem II and let

$$[\underline{v},\bar{v}] = \{w \in L^\infty(\Omega) : \underline{v} \leq w \leq \bar{v} \text{ a.e. on } \Omega\} \quad .$$

Proposition 7 Let $u_0 \in [\underline{v},\bar{v}]$ satisfy (1.13). Then

(i) $u(t,u_0) \in [\underline{v},\bar{v}]$ for $t > 0$

and

(ii) $\omega(u_0) \subset [\underline{v},\bar{v}] \cap X$.

Proof. It follows from the definitions that $\underline{v}(\bar{v})$ is a time-independent subsolution (supersolution) of Problem I^*. Hence Theorems 5 and 6 imply (i). The assertion (ii) follows from Theorem 6 and the fact that $[\underline{v},\bar{v}] \cap X$ is closed in X.

Corollary 8. If $u_0 \in [\underline{v},\bar{v}]$ satisfies (1.13) and $[\underline{v},\bar{v}] \cap E = \{g\}$ is a singleton, then $u(t,u_0) \to g$ in X as $t \to \infty$.

We next give three applications of this Corollary to the determination of domains of attraction of the various isolated elements of $E(L)$.

(1.15) Let $L \in (L_0,L_1]$. Choose $\ell \in [L_0,L)$ and $\xi \in (-L,L)$ such that $-L < \xi - \ell$, $\xi + \ell < L$. Set

$$\underline{v}(x) = \begin{cases} p(x-\xi,\ell) & \text{for } x \in [\zeta-\ell,\xi+\ell] \quad , \\ 0 & \text{for } x \notin [\zeta-\ell,\xi+\ell] \quad . \end{cases}$$

Then $\underline{v}$ is a subsolution of Problem II. Clearly $\bar{v} \equiv 1$ is a supersolution. Since $\ell \in [L_0,L)$,

$$\underline{v}(\xi) = p(0,\ell) > p(0,L) \geq p(\xi,L)$$

(see Figure 3(a) below) and it follows that

$$[\underline{v},\bar{v}] \cap E(L) = \{q(L)\} \quad .$$

It now follows from Corollary 8 that $u(t,u_0) \to q(L)$ in X as $t \to \infty$ whenever $u_0 \in [\underline{v},\bar{v}]$.

(1.16) Let $L \varepsilon [L_0, L_1)$. Chose $\ell \varepsilon (L, L_1]$ and $\eta \varepsilon (-L, L)$ such that $\eta - \ell < -L$, $L < \eta + \ell$, and let

$$\bar{w}(x) = p(x-\eta, \ell) \quad .$$

Then $\bar{w}$ is a supersolution of Problem II. Since $\underline{w} = 0$ is a subsolution, $[\underline{w}, \bar{w}]$ is invariant. In this case (see Figure 3(b))

$$\bar{w}(0) = p(-\eta, \ell) < p(0, \ell) < p(0, L) \quad .$$

Hence

$$[\underline{w}, \bar{w}] \cap E(L) = \{0\}$$

and therefore $u(t, u_0) \to 0$ as $t \to \infty$ in X when $u_0 \varepsilon [\underline{w}, \bar{w}]$.

(1.17) Let $L > L_1$. For $x_1, x_2 \varepsilon (-L, L)$ such that

$$-L < x_1 - L_1 < x_2 - L_1 < x_1 + L_1 < x_2 + L_1 < L$$

define

$$v^+(x) = \max(p(x-x_1, L_1), p(x-x_2, L_1))$$

$$v^-(x) = \min(p(x-x_1, L_1), p(x-x_2, L_1)) \quad .$$

See Figure 4. It is easy to verify that v^+ is a subsolution and v^- is a supersolution of Problem II. Moreover,

$$[v^+, 1] \cap E(L) = \{q(L)\}, \ [0, v^-] \cap E(L) = \{0\} \quad .$$

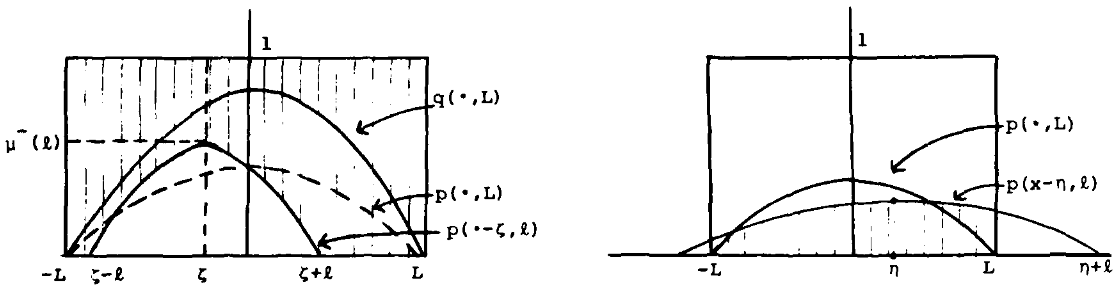
Thus

$$\lim_{t \to \infty} u(t, u_0) = q(L) \quad \text{for all} \quad u_0 \ \varepsilon \ [v^+, 1]$$

and

$$\lim_{t \to \infty} u(t, u_0) = 0 \quad \text{for all} \quad u_0 \ \varepsilon \ [0, v^-] \quad .$$

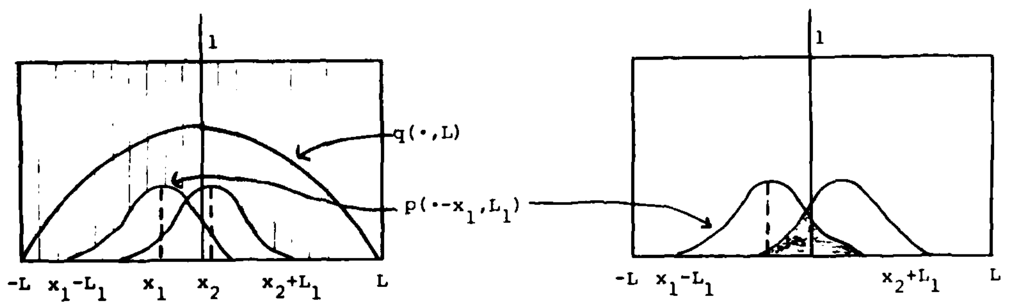
Figure 3



(a) Domain of attraction for $q(\cdot,L)$ where $L \in (L_0,L_1]$

(b) Domain of attraction for 0 when $L \in (L_0,L_1]$

Figure 4



(a) Domain of attraction for $q(\cdot,L)$ when $L > L_1$

(b) Domain of attraction for 0 when $L > L_1$

It is clear that more elaborate domains of attraction for q and 0 can be constructed. We leave further constructions to the interested reader.

Part II. General Theory

In this part of the paper we first prove existence, uniqueness and continuous dependence on initial data of solutions of Problem I of the introduction together with comparison results. These results are all more or less known in various contexts, but the presentation here collects them quite conveniently. (See the remarks at the end of this section.) After this the stabilization result used in Part I is proved.

Section 3. A Preliminary Estimate

Consider the problem

$$(III)\qquad \begin{cases} u_t = \eta(u)_{xx} + g(x,t) & (x,t) \in Q\ , \\ u(\pm L,t) = \psi_{\pm}(t) & t \in (0,\infty)\ , \\ u(x,0) = u_0(x) & x \in \Omega \end{cases}$$

where we assume that η and the data g, $\psi_{\pm}$ and u_0 satisfy the following set of assumptions:

A1. $\eta : R \to R$ is locally Lipschitz continuous and nondecreasing,

A2. $g \in L^1(Q_T)$ for each $T > 0$,

A3. $\psi_{\pm} \in L^1_{\ell oc}([0,\infty])$,

A4. $u_0 \in L^{\infty}(\Omega)$.

These will be called "assumption A".

Definition. A solution u of Problem III on $[0,T]$ is a function u with the following properties:

(i) $u \in C([0,T] : L^1(\Omega)) \cap L^{\infty}(Q_T)$,

(ii) $\int_\Omega u(t)\varphi(t) - \iint_{Q_t} (u\varphi_t + \eta(u)\varphi_{xx}) +$

$$\int_0^t \eta(\psi_+(s))\varphi_x(L,s) - \eta(\psi_-(s))\varphi_x(-L,s)ds$$

$$= \int_\Omega u_0\varphi(0) + \iint_{Q_t} g\varphi$$

for all $\varphi \in C^2(\bar{Q}_T)$ such that $\varphi \geq 0$, $\varphi = 0$ at $x = \pm L$ and $0 \leq t \leq T$. A solution on $[0,\infty)$ means a solution on each $[0,T]$, and a subsolution (supersolution) is defined by (i), and (ii) with equality replaced by $\leq$ ($\geq$).

<u>Proposition 9</u> Let $\hat{u}$ be a supersolution of Problem III on $[0,T]$ with data $\hat{g}$, $\hat{u}_0$, $\hat{\psi}_\pm$ and u be a subsolution on $[0,T]$ with data g, u_0, $\psi_\pm$ all satisfying assumption A. Then if $\psi_\pm \leq \hat{\psi}_\pm$ we have for each $\lambda > 0$ and $0 \leq t \leq T$

$$e^{\lambda t} \int_\Omega (u(t) - \hat{u}(t))^+ \leq \int_\Omega (u_0-\hat{u}_0)^+ + \iint_{Q_t} e^{\lambda s}(g-\hat{g} + \lambda(u-\hat{u}))^+ \tag{2.1}$$

where $r^+ = \max(r,0)$.

<u>Proof</u>. Since $\hat{u}$ (u) is a supersolution (subsolution) and $\varphi_x \geq 0$ (≤ 0) at $x = -L(+L)$, we find, using $\eta(\hat{\psi}_\pm) \geq \eta(\psi_\pm)$ (by A1)

$$\begin{aligned} &\int_\Omega (u(t) - \hat{u}(t))\varphi(t) - \iint_{Q_t} (u - \hat{u})(\varphi_t + a\varphi_{xx}) \leq \\ &\leq \int_\Omega (u_0 - \hat{u}_0)\varphi(0) + \iint_{Q_t} (g - \hat{g})\varphi \quad , \end{aligned} \tag{2.2}$$

where $a = (\eta(u) - \eta(\hat{u}))/(u - \hat{u})$ for $u \neq \hat{u}$ and $a = 0$ otherwise, for all $\varphi \in C^2(\bar{Q}_T)$ such that $\varphi \geq 0$ and $\varphi = 0$ at $x = \pm L$ and $0 \leq t \leq T$. By A1 and the boundedness of u, $\hat{u}$, we have $a \in L^\infty(Q_T)$ and $a \geq 0$.

We now construct a special sequence of functions $\{\varphi_n\}$ *to use in* (2.1). Fix $T > 0$ and choose a sequence $\{a_n\}$ of smooth functions such that

$$\frac{1}{n} \leq a_n \leq \|a\|_{L^\infty(Q_T)} + \frac{1}{n}$$

and

$$(a_n - a)/\sqrt{a_n} \to 0 \quad \text{in} \quad L^2(Q_T) \quad .$$

This is easily seen to be possible. Next let $\chi \in C_0^\infty(\Omega)$ be such that $0 \leq \chi \leq 1$. Finally let φ_n be the solution of the backward problem

$$(2.3)\qquad \begin{cases} \varphi_{nt} + a_n \varphi_{nxx} = \lambda \varphi_n & \text{for } x \in \Omega,\ t \in [0,T) \quad , \\ \varphi_n(\pm L, t) = 0 & \text{for } t \in [0,T) \quad , \\ \varphi_n(x,T) = \chi(x) & \text{for } x \in \Omega \quad . \end{cases}$$

This is a nondegenerate parabolic problem and has a unique solution $\varphi_n \in C^\infty(\bar{Q}_T)$.

Lemma 10 The function φ_n has the following properties:

(i) $0 \leq \varphi_n \leq e^{\lambda(t-T)}$ on $\bar{Q}_T$,

(ii) $\iint\limits_{Q_T} a_n(\varphi_{nxx})^2 \leq c$,

(iii) $\sup\limits_{0 \leq t \leq T} \int_\Omega (\varphi_{nx})^2(t) \leq c$

where the constant c depends only on χ.

Proof. Part (i) is immediate from the maximum principle and $0 \leq \chi \leq 1$. To prove (ii) and (iii), multiply the equation solved by φ_n by φ_{nxx} and integrate over $\Omega \times (t,T)$ to find - after an integration by parts -

$$-\int_t^T \int_\Omega \varphi_{nx}\varphi_{nxt} + \int_t^T \int_\Omega a_n(\varphi_{nxx})^2 = -\lambda \int_t^T \int_\Omega (\varphi_{nx})^2$$

or

$$\frac{1}{2}\int_\Omega (\varphi_{nx})^2(t) + \int_t^T \int_\Omega a_n(\varphi^n_{xx})^2 + \lambda \int_t^T \int_\Omega (\varphi_{nx})^2 = \frac{1}{2}\int_\Omega (\chi_x)^2$$

from which we have the desired estimates.

If we set $t = T$ and $\varphi = \varphi_n$ in (2.2) we obtain

$$(2.4)\qquad \int_\Omega (u(T) - \hat u(T))\chi - \iint_{Q_T} (u-\hat u)(a-a_n)\varphi_{nxx} \le$$

$$\le \int_\Omega (u_0 - \hat u_0)\varphi_n(0) + \iint_{Q_T} (g-\hat g + \lambda(u-\hat u))\varphi_n$$

$$\le \int_\Omega (u_0 - \hat u_0)^+ e^{-\lambda T} + \iint_{Q_T} (g-\hat g + \lambda(u-\hat u))^+ e^{\lambda(s-T)} \quad .$$

Since

$$\iint_{Q_T} |a-a_n|\ |\varphi_{nxx}| = \iint_{Q_T} \frac{|a-a_n|}{\sqrt{a_n}} (\sqrt{a_n}|\varphi_{nxx}|)$$

we have, by Lemma 10 (ii),

$$\|(a-a_n)\varphi_{nxx}\|_{L^1} \le \left\|\frac{(a-a_n)}{\sqrt{a_n}}\right\|_{L^2} \|\sqrt{a_n}\ \varphi_{nxx}\|_{L^2}$$

$$\le c^{1/2} \left\|\frac{(a-a_n)}{\sqrt{a_n}}\right\|_{L^2}$$

which tends to zero as $n \to \infty$ by the choice of a_n. (The spaces L^1 and L^2 here are taken over Q_T.) Thus, letting $n \to \infty$ in (2.4) we obtain

$$(2.5)\qquad \int_\Omega (u(T) - \hat u(T))\chi \le \int_\Omega (u_0 - u_0)^+ e^{-\lambda T} + \iint_{Q_T} (g-\hat g + \lambda(u-\hat u))^+ c^{\lambda(s-T)} \quad .$$

This inequality holds for every $\chi \in C_0^\infty(\Omega)$ with $0 \le \chi \le 1$. Hence it continues to hold for $\chi(x) = 1$ on $\{x : u(T) > \hat u(T)\}$ and $\chi = 0$ otherwise (i.e. $\chi = \operatorname{sign}(u(T)-\hat u(T))^+$, completing the proof. (Clearly T may be replaced by any t, $0 \le t \le T$ in the argument.)

<u>Corollary 11</u>. Let u and $\hat u$ be solutions of Problem III with data g, u_0, $\psi_\pm$ and $\hat g$, $\hat u_0$, $\psi_\pm$. Then

$$(2.6)\qquad \|u(t) - \hat u(t)\|_{L^1(\Omega)} \le \|u_0 - \hat u_0\|_{L^1(\Omega)} + \int_0^t \|g(s) - \hat g(s)\|_{L^1(\Omega)}\, ds \quad .$$

Thus, in particular, solutions of Problem III are unique.

Proof. Setting $\lambda = 0$ in (2.1) we obtain

$$\int_\Omega (u(t) - \hat{u}(t))^+ \leq \int_\Omega (u_0 - \hat{u}_0)^+ + \iint_{Q_t} (g - \hat{g})^+$$

and

$$\int_\Omega (\hat{u}(t) - u(t))^+ \leq \int_\Omega (\hat{u}_0 - u_0)^+ + \iint_{Q_t} (\hat{g} - g)^+ \quad .$$

Adding these estimates yields (2.6).

Uniqueness and continuous dependence for Problem II.

We now return to Problem I. Here we take $\eta(u) = |u|^{m-1}u$ which satisfies A1. By a solution of Problem I we mean a solution u of Problem III with $g = f(u)$, $\psi_\pm = 0$ and so on for sub-and supersolutions. Since solutions are bounded by definition, $g \in L^\infty(Q_T) \subset L^1(Q_T)$, and we may use the previous results.

Theorem 12

(i) Let $u, \hat{u}$ be solutions of Problem I on $[0,T]$ with initial data u_0 and $\hat{u}_0$ respectively. Let K be a Lipschitz constant for f on $[-M,M]$ where $M = \max\left(\|u\|_{L^\infty(Q_T)}, \|\hat{u}\|_{L^\infty(Q_T)}\right)$. Then

$$\|u(t) - \hat{u}(t)\|_{L^1(\Omega)} \leq e^{Kt} \|u_0 - \hat{u}_0\|_{L^1(\Omega)} \quad .$$

(ii) Let u be a subsolution and $\hat{u}$ a supersolution of Problem I with initial data u_0 and $\hat{u}_0$. Then if $u_0 \leq \hat{u}_0$ it follows that

$$u \leq \hat{u} \quad .$$

Proof. With the assumptions of (ii), Proposition 7 yields

$$(2.7) \qquad e^{\lambda t} \int_\Omega (u(t) - \hat{u}(t))^+ \leq \int_\Omega (u_0 - \hat{u}_0)^+ + \int_0^t \int_\Omega e^{\lambda s}(f(u) - f(\hat{u}) + \lambda(u - \hat{u}))^+ \quad .$$

Set $\lambda = K$ (defined as in (i)). Then $r \to f(r) + Kr$ is nondecreasing on $[-M,M]$ and

$$(f(u) - f(\hat{u}) + K(u - \hat{u}))^+ \leq 2K(u - \hat{u})^+ \quad .$$

Thus if we write

$$h(t) = e^{Kt} \int_\Omega (u(t) - \hat{u}(t))^+$$

(2.7) implies

$$h(t) \leq h(0) + 2K \int_0^t h(s)ds$$

which implies, by Gronwall's lemma, that $h(t) \leq h(0)e^{2Kt}$ or

$$\int_\Omega (u(t) - \hat{u}(t))^+ \leq e^{Kt} \int_\Omega (u_0 - \hat{u}_0)^+ \quad .$$

This proves (ii). The assertion of (i) follows by adding the corresponding inequality for $(\hat{u} - u)^+$.

Remark. The proofs here are correct for any η satisfying A1 and do not require $0 \leq u_0, \hat{u}_0 \leq 1$.

Section 4 Existence

We begin by regularizing the problem. Let $\varepsilon > 0$ and consider

$$(I_\varepsilon) \qquad \begin{cases} u_t = (u^m)_{xx} + f_\varepsilon(u) & (x,t) \in Q_T \\ u(\pm L,t) = \varepsilon & t \in (0,T] \\ u(x,0) = u_0(x) + \varepsilon & x \in \bar{\Omega} \end{cases}$$

where

$$f_\varepsilon(u) = f(u - \varepsilon) \quad .$$

The properties of f and u_0 we will use are:

(H) $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous, $f(0) = f(1) = 0$

and $u_0 \in L^\infty(\Omega)$, $0 \le u_0 \le 1$,

which we refer to as hypotheses H. For a while we also assume $u_0 \in C_0^\infty(\Omega)$. Then Problem I_ε has, by classical results, a unique solution $u_\varepsilon \in C^{2,1}(\bar{Q}_T)$ and

(2.8) $$\varepsilon \le u_\varepsilon \le 1 + \varepsilon \text{ in } \bar{Q}_T \quad .$$

Multiplying the equation of I_ε by $(u_\varepsilon^m)_t$ and performing obvious manipulations yields

(2.9)
$$\frac{4m}{(m+1)^2} \int_\tau^t \int_\Omega \left(u_\varepsilon^{\frac{m+1}{2}}\right)_t^2 + \frac{1}{2} \int_\Omega \{(u_\varepsilon^m)_x(t)\}^2 - m \int_\Omega F_\varepsilon(u_\varepsilon)(t)$$
$$= \frac{1}{2} \int_\Omega \{(u_\varepsilon^m)_x(\tau)\}^2 - m \int_\Omega F_\varepsilon(u_\varepsilon)(\tau)$$

for $0 \le \tau \le t \le T$, where

$$F_\varepsilon(u) = \int_\varepsilon^u S^{n-1} f_\varepsilon(s)ds \quad .$$

In particular, putting $\tau = 0$,

(2.10) $$\int_0^T \int_\Omega \left(u_\varepsilon^{\frac{m+1}{2}}\right)_t^2, \sup_{0<t<T} \int_\Omega \{(u_\varepsilon^m)_x(t)\}^2 \le K$$

where K depends on f, $\int_\Omega (u_0^m)_x^2$ but not on $\varepsilon \in (0,1)$ or T. Set $v_\varepsilon = u_\varepsilon^m$. Then (2.9), (2.10) imply

$$
(2.11)\qquad \begin{cases} \varepsilon^m \leq v_\varepsilon \leq (1+\varepsilon)^m \\ \|v_{\varepsilon x}(t)\|_{L^2(\Omega)} \leq \sqrt{K} \\ \|v_{\varepsilon t}\|^2_{L^2(0,T:L^2(\Omega))} = \int_0^T \int_\Omega \{((u_\varepsilon^{\frac{m+1}{2}})^{\frac{2m}{m+1}})_t\}^2 \end{cases}
$$

$$
\leq \left(\frac{2m}{m+1}\right)^2 (1+\varepsilon)^{m-1} \int_0^T \int_\Omega (u^{\frac{m+1}{2}})_t^2
$$

$$
\leq \left(\frac{2m}{m+1}\right)^2 (1+\varepsilon)^{m-1} K \quad .
$$

It follows from (2.11) that $\{v_\varepsilon\}$, $0 < \varepsilon < 1$ is equicontinuous from $[0,T]$ into $L^2(\Omega)$ with values in a bounded subset of $H^1(\Omega)$ (which is compactly imbedded in $L^2(\Omega)$). Hence, by Arzela-Ascoli's Theorem, there is a $v \in C([0,T]: L^2(\Omega))$ and $\varepsilon_n \to 0$ such that $v_{\varepsilon_n} \to v$ in $C([0,T]: L^2(\Omega))$. Then $u_{\varepsilon_n} \to v^{1/m} \equiv u$ and $u^m_{\varepsilon_n} \to u^m$ in $C([0,T]: L^2(\Omega))$. It is very simple to show that u is a solution of Problem I, and we omit this. (Note that $C([0,T] : L^2(\Omega)) \subset C([0,T] : L^1(\Omega))$.)

It remains to remove the restriction $u_0 \in C_0^\infty(\Omega)$. To this end, let u_0 satisfy (2.8) and choose a sequence $\{u_{0n}\} \subset C_0^\infty(\Omega)$, $0 \leq u_{0n} \leq 1$, such that

$$
(2.12)\qquad \|u_0 - u_{0n}\|_{L^1(\Omega)} \to 0 \quad \text{as} \quad n \to \infty \quad .
$$

Let u_n be the solution of Problem I with initial data u_{0n}. According to Theorem 12(i)

$$
(2.13)\qquad \sup_{0\leq t\leq T} \|u_j(t) - u_k(t)\|_{L^1(\Omega)} \leq e^{KT} \|u_{0j} - u_{0k}\|_{L^1(\Omega)}
$$

where K is now a Lipschitz constant for f on $[0,T]$. By $0 \leq u_j \leq 1$, (2.12), (2.13) there is a $u \in C([0,T] : L^1(\Omega))$, $0 \leq u \leq 1$, such that $u_j \to u$ in $C([0,T] : L^1(\Omega))$. Clearly u is a solution of Problem I and so we have proved:

Theorem 13. Let hypotheses H hold. Then Problem I has a unique solution u. Moreover, $0 \le u \le 1$.

Section 5. Regularization

In this section we shall prove a regularizing property of the solution operator of Problem I.

Theorem 14. Let hypotheses H hold and u be the solution of Problem I. Then for each $\tau > 0$ there is a constant M_τ, independent of u_0, such that

(i) $u^m(t)_x \in L^\infty(\Omega)$ for $t > \tau$

and

$$\text{(ii)} \quad \begin{cases} \|u^m(t)_x\|_{L^\infty(\Omega)} \le M_\tau \text{ and essential variation } u^m(t)_x \le M_\tau \\ \text{for } t > \tau . \end{cases}$$

Proof. Following [6] we denote the solution operator of Problem III with $\eta(r) = |r|^{m-1}r$ and $\psi_\pm = 0$ by $S(t,u_0,g)$ - that is, $S(t,u_0,g)$ is the solution of Problem III at time t if this problem has a solution. By Corollary 11, S has the Properties

$$\|S(t,u_0,g) - S(t,\hat{u}_0,\hat{g})\|_{L^1} \le \|u_0 - \hat{u}_0\|_{L^1} + \int_0^t \|g(s) - \hat{g}(s)\|_{L^1} ds \tag{2.14}$$

and

$$\lambda^{\frac{1}{m-1}} S(\lambda t,u_0,g) = S(t, \lambda^{\frac{1}{m-1}} u_0, \lambda^{\frac{m}{m-1}} g_\lambda), \ \lambda > 0 \tag{2.15}$$

where $g_\lambda(t)(\cdot) = g(\lambda t)(\cdot)$. To establish (2.15) one merely checks - using $\eta(r) = |r|^{m-1}r$ - that $\lambda^{\frac{1}{m-1}} S(\lambda t,u_0,g)$ is indeed a solution of Problem III with data $\lambda^{\frac{1}{m-1}}u_0$, $\lambda^{\frac{1}{m-1}}g_\lambda$ in place of u_0,g and invokes the uniqueness. Now the solution $u(t,u_0)$ of Problem I is exactly a solution of $u = S(t,u_0,f(u))$. By Theorem 7 of [6], properties (2.14) and (2.15) of S and the Lipschitz continuity of f imply that for $\tau > 0$, $0 < h \le \tau$, $t \ge 0$

$$\frac{1}{h}\,\|u(t+\tau+h,u_0) - u(t+\tau,u_0)\|_{L^1} = \frac{1}{\tau}\left(\frac{\tau}{h}\,\|u(\tau+h,u(t,u_0)) - u(\tau,u(t,u_0))\|_{L^1}\right.$$

$$\leq \frac{1}{\tau}\,E\big(\tau,\ \|u(t,u_0)\|_{L^1}\big)$$

where E is a nondecreasing function of its arguments. Since $0 \leq u \leq 1$, we have $\|u(t,u_0)\|_{L^1} \leq \text{meas}\ \Omega = 2L$ and it follows that $\tau^{-1}E(\tau,2L)$ is a Lipschitz constant for $t \to u(t+\tau,u_0)$ on $[0,\infty)$.

The proof is completed by means of the follows lemma:

Lemma 15. Let $v(t)$ be Lipschitz continuous with constant L, and $w(t)$, $z(t)$ be continuous from $[0,\infty)$ into $L^1(\Omega)$ and

$$v_t = w_{xx} + z \quad \text{in} \quad \mathcal{D}'(\Omega) \quad .$$

Then $w(t)_x \in L^\infty(\Omega)$ for each t and

$$\text{(2.16)} \qquad \text{essential variation}\ \ w(t)_x \leq L + \|z(t)\|_{L^1} \quad .$$

We apply this lemma to the equation

$$u_t = (u^m)_{xx} + f(u)$$

which holds in $\mathcal{D}'$ (i.e., in the sense of distributions). As shown above, $t \to u(t,u_0)$ is Lipschitz continuous from $[\tau,\infty)$ into $L^1(\Omega)$ with a constant L_τ independent of u_0. By Lemma 15 $u^m(t)_x \in L^\infty(\Omega)$ for $t \geq \tau > 0$ and the variation of $u^m(t)_x$ is bounded by $L_\tau + \|f(u(t))\|_{L^1}$, which is bounded. If $v : [0,1] \to \mathbb{R}$ is smooth then $v_x(a) = v(1) - v(0)$ for some $a \in (0,1)$ and then

$$\|v_x\|_{L^\infty} \leq |v_x(a)| + \text{variation}\ \ v_x$$

$$\leq 2\|v\|_{L^\infty} + \text{variation}\ \ v_x \quad .$$

By approximation with smooth functions we conclude

$$\|(u^m)_x\|_{L^\infty} \leq 2\|u^m\|_{L^\infty} + \text{ess variation}\ \ (u^m)_x$$

$$\leq 2 + \text{ess variation}\ \ (u^m)_x \quad .$$

Thus the assertions of the theorem are established.

It remains to prove the lemma.

Proof of Lemma 15. Define the averages

$$v_h = \frac{1}{h}\int_h^{t+h} v,\ w_h = \frac{1}{h}\int_t^{t+h} w,\ z_h = \frac{1}{h}\int_t^{t+h} z \quad .$$

Indeed, these averages can be defined for an arbitrary distribution F on Q by

$$F_h[\varphi] = F[\frac{1}{h}\int_{(t-h)}^{t}\varphi] \quad \text{for} \quad \varphi \in C_0^\infty(Q)$$

and then it is easily checked that the operation commutes with differentiations. Hence $h^{-1}(v(t) - v(t-h)) = v_{ht} = w_{hxx} + z_h$. We conclude that $w_{hxx} \in L^1$ and

$$\text{variation } w_{hx} = \|w_{hxx}\|_{L^1} \le L + \|z_h(t)\|_{L^1}$$

$$\to L + \|z(t)\|_{L^1} \quad \text{as} \quad h \to 0 \quad .$$

Since w_h, w_{hxx} remain bounded in $L^1(\Omega)$, w_{hx} is bounded in $L^\infty(\Omega)$ and $w_{hx} \to w_x$ in $L^p(\Omega)$, $1 \le p < \infty$. Letting $h \to 0$ and using the lower semicontinuity of the variation we obtain (2.16).

Section 6. Stabilization

Let $0 \le u_0 \le 1$ and $u = u(t,u_0)$ be the solution of Problem I emanating from u_0. For each $\tau > 0$ define the semiorbit

$$\gamma_\tau(u_0) = \{u(t,u_0) : t \ge \tau\} \quad .$$

According to Theorem 14, $\gamma_\tau(u_0) \subset X_\tau$ where X_τ is the complete metric space consisting of those $w \in L^\infty(\Omega)$ such that

$$(2.17) \qquad \begin{cases} 0 \le w \le 1,\ (w^m)_x \in L^\infty(\Omega),\ \|(w^m)_x\|_{L^\infty} \le M_\tau \\ \\ \text{and essential variation } (w^m)_x \le M_\tau \quad , \end{cases}$$

where M_τ is as in Theorem 14, equipped with the metric

$$d(u,v) = \|u-v\|_{L^1(\Omega)} + \|(u^m - v^m)_x\|_{L^2(\Omega)} \quad . \tag{2.18}$$

One easily checks that X_τ is complete. Moreover, X_τ is compact, Indeed, $\{w^m : w \in X_\tau\}$ is bounded in, e.g., $W^{1,\infty}(\Omega)$ and is thus precompact in $L^1(\Omega)$. It follows that $\{w : w \in X_\tau\}$ is precompact in $L^1(\Omega)$. Similarly, $\{(w^m)_x : w \in X_\tau\}$ is bounded in $L^\infty(\Omega)$ and in variation. Thus it is precompact in $L^1(\Omega)$ and then, by the L^∞ boundedness, in every $L^p(\Omega)$, $1 \leq p < \infty$. The compactness of X_τ follows. We also let

$$X = \{u \in L^\infty(\Omega) : 0 \leq u \leq 1,\ (u^m)_x \in L^2(\Omega)\} \tag{2.19}$$

equipped with the metric (2.18). Observe that

$$\{u_n\} \subset X_\tau \quad \text{and} \quad \|u_n - u\|_{L^1(\Omega)} \to 0 \implies u \in X_\tau \quad \text{and} \quad d(u_n,u) \to 0 \quad . \tag{2.20}$$

This is the standard remark that weakening a metric of a compact metric space produces the same topology.

To study the large time behaviour of $u(t,u_0)$ we introduce its ω-limit set: $\omega(u_0) = \{w \in X : u(t_n,u_0) \to w$ in X for some sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty\}$. We collect some basic remarks.

<u>Proposition 16</u>. Let hypotheses H hold. Then

(i) $\gamma_\tau(u_0)$ is a precompact subset of X for $\tau > 0$.

(ii) $u(\cdot,u_0) \in C((0,\infty) : X)$.

(iii) $\omega(u_0)$ is nonempty and connected in X.

(iv) If $w \in \omega(u_0)$, then $u(t,w) \in \omega(u_0)$ for $t > 0$.

Proof. Since $\gamma_\tau \subset X_\tau$ which is compact, (i) follows. By (2.20), $t \to u(t,u_0)$ is continuous into X_τ on $t > \tau$ if and only if it is continuous into $L^1(\Omega)$, whence we have (ii). The assertion (iii) follows at once from (i) and (ii). For (iv), we use that $u(t+t_n,u_0) = u(t,u(t_n,u_0))$ so if $u(t_n,u_0) \to w$ in X (and so in $L^1(\Omega)$) we have $u(t+t_n,u_0) \to u(t,w)$ in L^1 (and hence in X) by Theorem 12. Thus $u(t,w) \in \omega(u_0)$.

Next consider the function $V : X \to R$ given by

$$V(\zeta) = \int_\Omega (1/2\,(\zeta^m)'^2 - mF(\zeta))dx$$

where

$$F(r) = \int_0^r \rho^{m-1} f(\rho)d\rho \quad .$$

Clearly $V : X \to R$ is continuous.

Lemma 17. Let hypotheses H be satisfied. Then

$$(u^{\frac{m+1}{2}})_t \in L^2_{\ell oc}(0,\infty : L^2(\Omega)) \quad \text{and}$$

$$\frac{4m}{(m+1)^2} \int_s^t \int_\Omega \{(u^{\frac{m+1}{2}})_t\}^2 + V(u(t,u_0)) \le V(u(s,u_0)) \quad \text{for} \quad t > s > 0 \quad .$$

The proof is postponed briefly while we establish the next result.

Theorem 18. Let hypotheses H be satisfied. Then $\omega(u_0) \subset E$.

Proof. By Lemma 17 $t \to V(u(t,u_0))$ is nonincreasing on $t > 0$. Since V is continuous on X this implies

$$V(w) = \inf_{t>0} V(u(t,u_0)) \equiv V_\infty \quad \text{for} \quad w \in \omega(u_0) \quad .$$

By the invariance of $\omega(u_0)$, $V(u(t,w)) = V_\infty$ for $t > 0$, $w \in \omega(u_0)$. Combining this with Lemma 17 we deduce that $\left(u(t,w)^{\frac{m+1}{2}}\right)_t \equiv 0$ and thus $u(t,w) \equiv w$. The definition of a solution of Problem I then implies

$$\int_\Omega (w^m \varphi_{xx} + f(w)\varphi) = 0$$

whenever $\varphi \in C^2(\bar{\Omega})$, $\varphi > 0$ and $\varphi(\pm L) = 0$. But this implies $(w^m)_{xx} + f(w) = 0$ in $\mathcal{D}'$. Since $w \in L^\infty$ and f is Lipschitz, the equation holds classically (i.e., $w^m \in C^2(\bar{\Omega})$) and $w = 0$ at $\pm L$. Hence $w \in E(L)$.

<u>Proof of Lemma 17</u>. We go back into the existence proof where, assuming $u_0 \in C_0^\infty(\Omega)$, we constructed smooth solutions u_ε of approximate problems I_ε. Putting $\tau = 0$ in (2.9) and letting ε tend to zero through the sequence ε_n as in that proof yields

$$\text{(2.21)} \qquad \frac{4m}{(m+1)^2}\int_0^t \int_\Omega \{(u^{\frac{m+1}{2}})_t\}^2 + V(u(t,u_0)) < V(u_0) \quad .$$

We note also that since $u_\varepsilon^m - \varepsilon^m \in H_0^1(\Omega)$ for $t > 0$ and (2.10) holds, we have $u^m \in H_0^1(\Omega)$ for $t > 0$. To deduce (2.21) for general $u_0 \in H_0^1(\Omega)$, choose $\{u_{0n}\} \subset C_0^\infty(\Omega)$ so that $u_{0n} \to u_0$ in X as $n \to \infty$. Writing (2.21) for $u_n = u(t,u_{0n})$ in place of u and letting $n \to \infty$ establishes the inequality. The lemma now follows if we show $u(s,u_0)^m \in H_0^1(\Omega)$ for $s > 0$. But we know $u(s,u_0)^m \in H_0^1(\Omega)$ for $u_0 \in H_0^1(\Omega)$ and that $(u(s,u_0)^m)_x$ is bounded in $L^2(\Omega)$ (even $L^\infty(\Omega)$) independently of u_0, $0 < u_0 < 1$, whence the result.

<u>Remarks on Part II</u>

<u>Section 3</u>. The reader should notice that the whole development of this section is valid if Ω is a domain in $\mathbb{R}^N$ rather than an interval of $\mathbb{R}$. For this one replaces $\eta(u)_{xx}$ by $\Delta\eta(u)$ (or $E\eta(u)$ where E is a suitable elliptic operator) and modifies the statement

of the boundary conditions in Problem III. The estimate (2.1) remains valid as stated. Various of the ideas used in this proof occur for example, in [14], [15], [17]. The paper [19] is an early example of L^1-type estimates.

It is known in nonlinear semigroup theory that Problem III (with $\Omega \subseteq R^N$) has a unique solution in the sense of this section if $\psi_\pm = 0$ and $g \in L^\infty(Q_T)$. Moreover, for this η need only be continuous (not locally Lipschitz continuous) and the estimates (2.1) and (2.6) are valid for these solutions (with $\psi_\pm = 0$). However, the proof goes by showing uniqueness (without establishing the estimates) and obtaining the estimates in the construction of the solutions. See [6] and its references concerning the uniqueness. See, e.g., [1], [8], [12] concerning the semigroup theory. One will not find the claims above presented clearly in these sources, they are (true) folk-lore. Moreover, the semigroup theory provides solutions to Problem III (with $\Omega \subset R^N$, $\psi_\pm = 0$) if g is merely L^1 and u_0 is merely L^1 and η need not be a function but a graph. One can also take $u_0 \in H^{-1}(\Omega)$, $f \in L^1(0,T : H^{-1}(\Omega))$. However, one does not then use the above notion of solution - in particular, u need not be bounded. See [5] concerning the H^{-1} theory. Similar remarks pertain to Problem I, although now it is the L^1 semigroup theory (rather than H^{-1}) which should be used.

Section 4. We have given the quickest existence proof suitable for our purpose. It is rather standard and restrictive in that it requires $u_0 > 0$. Alternatives are provided by the semigroup theory (see above), but this is clumsy as regards approximations by smooth functions. To allow u_0 to change sign and η to be less regular, one can approximate η by smooth η_ε and regularize the equation by $u_t = \Delta(\eta_\varepsilon(u) + \varepsilon u) + f(u)$. (Again the proofs in this section work for $\Omega \subseteq R^N$).

The argument for taking the limit in this section works in essence if $|u|^{m-1}u$ is replaced by $\eta(u)$ where η is strictly increasing and either η or η^{-1} is Lipschitz continuous. For general η, the semigroup theory is best. With it one has that

$$\begin{cases} u_t = \Delta\eta(u) + f(u) & \text{in } Q_T , \\ u|_{\partial\Omega} = 0 & t > 0 , \\ u|_{t=0} = u_0 & \text{in } \Omega , \end{cases}$$

has a solution (in the semigroup sense) under very mild conditions which we do not detail here. See also [11], [3].

Section 5. The relevant reference here has been given - [6]. This section depends on $N = 1$ in that an estimate of Δu^m in the space of measures provides compactness of ∇u^m in L^2 only if $N = 1$. Moreover, we use strongly the special form $|u|^{m-1}u$ of the nonlinearity. The only related work on more general nonlinearities we know of is [9], [10].

Section 6. Owing to the remarks concerning Section 5, these arguments do not adapt to $N > 1$. Stabilization results in higher dimensions remain an interesting open problem.

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The central results concern the initial-value problem for $u_t = (u^m)_{xx} + u(1-u)(u-a)$, $-L \le x \le L$ and $t \ge 0$, under Dirichlet conditions at $x = \pm L$. Here $m > 1$ and $0 < a < (m+1)/(m+3)$. The equilibrium solutions of this problem are determined for each $L > 0$ and it is shown that the ω-limit set $\omega(u_0)$ of an initial datum u_0 with values in $[0,1]$ consists of a connected set of equilibria. This is used to determine some domains of attraction of isolated

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ABSTRACT (continued)

equilibria. A novel feature of the results is that for large L there are multiple parameter families of equilibria.

A second part of the paper gives a self-contained development of existence, uniqueness, maximum principles, and continuous dependence on data for more general equations $u_t = \eta(u)_{xx} + f(u)$. The results are employed in proofs of some of the theorems referred to above.

Interest in these questions is stimulated by the occurrence of such models in science, e.g. in fluid flow in porous media and biology.

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